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Factorization method and singular Hamiltonians

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Abstract. The oscillator singular supersymmetric partners are studied in detail, and their relationship with the equivalent problem of an oscillator plus an infinite barrier is clearly established. Interesting potentials, with possible applications to different physical models, appear as a bonus.

1. Introduction

Since the beginning of quantum mechanics, there has been a permanent interest in establishing connections between different quantum systems. One of the main reasons is that as only a few problems are known to be analytically solvable, the existence of such connections would allow us to widen their applicability range. In this respect, we have the very useful factorization technique, introduced by Schrödinger in the early days of quantum mechanics, and used by himself to solve the harmonic oscillator, the hydrogen atom, and the Kepler motion in a hypersphere [1]. It is worth mentioning that, although in a different context, the basic ideas can be traced back to Darboux [2] and what it is now called the Darboux transformation. Using these methods, Infeld and Hull presented an exhaustive classification of the one-dimensional factorizable potentials [3].

In the 1980s the topic of factorization received renewed stimulus owing to the introduction by Witten of supersymmetry in the context of quantum mechanics [4]. Later on, Mielnik extended the factorization technique and proved that the class of possible supersymmetric potentials is wider than was initially supposed [5]. The work of different authors clarified the links between the ideas mentioned above: Darboux transformation, factorization and supersymmetry [6, 7]. These algebraic methods coming from factorization were applied to different interesting quantum mechanical models in statistical mechanics, atomic and nuclear physics [8]. In general, the determination of supersymmetric partners originates singular as well as nonsingular potentials. Although, in principle, the main interest was attracted by the latter ones, the singular Hamiltonians soon proved to be relevant also. For example, singular Hamiltonians are related to the supersymmetry breaking process [9, 10], and to the existence of negative energy states in some supersymmetric systems (double well potentials) [10, 11].

Until now, most discussions about singular Hamiltonians have been concerned with three-dimensional potentials with rotational symmetry. Therefore, after separation of variables, the relevant coordinate has a radial character, the interval $[0, +\infty)$ being its domain. In those cases a centrifugal potential $\nu(\nu+1)/r^2$ naturally arises, which is responsible for the singularity. Many papers have been devoted to studying the solutions when such a term is present. However, here we will address the problem of singularities in its more general form, i.e. given a nonsingular Hamiltonian H with a singular partner H_δ, the spectra of H and H_δ have nothing in common, at least for the bound states. Therefore, even if the spectrum of the initial H is known, it is of no help in computing the spectrum of its partner H_δ. In this paper we show how to handle this situation for any singular H_δ obtained by means of factorization.

We present the results for the simplest case, the one-dimensional harmonic oscillator, but the key ideas can be applied to the other relevant cases listed in [3]. The main result coming from our study is the following: the eigenvalues and eigenvectors of the singular potentials arising from the factorization method applied to the oscillator can be straightforwardly related to the potential described by the same oscillator plus an infinite barrier placed at the singularity point. Therefore, what we have achieved is: (1) to compute the spectrum and the eigenfunctions for the singular Hamiltonians H_δ, and (2) to find an adequate partner Hamiltonian H for each H_δ, so that the couple H, H_δ becomes isospectral.

The paper is organized as follows: in section 2 we present the solution of the aforementioned quantum mechanical problem, a harmonic oscillator plus an infinite barrier potential. Section 3 is devoted to a close analysis of the factorization, but taking care mainly of the singular cases. Two different factorizations that generate singular potentials are presented there and discussed in detail. Some final conclusions put an end to this paper.

2. The harmonic oscillator plus an infinite barrier

In this section we shall deal with the time-independent one-dimensional Schrödinger equation for the harmonic oscillator potential plus an infinite barrier. Although this problem cannot be solved analytically, it allows for a simple treatment that give us some qualitative features as well as an arbitrary degree of accuracy for their solutions. The eigenvalue equation takes the usual form

$$H\psi(z) \equiv \left(-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}z^2} + V(z)\right)\psi(z) = E\psi(z) \tag{2.1}$$

with the following potential

$$V(z) = \begin{cases} kz^2/2 & x \geqslant B \\ \infty & x < B \end{cases} \qquad B \in \mathbb{R}. \tag{2.2}$$

We can eliminate some of the parameters that appear by making the following substitutions

$$\omega = \sqrt{\frac{k}{m}} \qquad x = \sqrt{\frac{2\omega m}{\hbar}}z \qquad \epsilon = -\frac{E}{\hbar\omega} \tag{2.3}$$

$$\psi(z) = y(x) \qquad b = \sqrt{\frac{2\omega m}{\hbar}}B. \tag{2.4}$$

Hence the Schrödinger equation (2.1) is rewritten as

$$\frac{\mathrm{d}^2 y(x)}{\mathrm{d}x^2} - v(x)y(x) = \epsilon y(x) \tag{2.5}$$

$$v(x) = \begin{cases} x^2/4 & x \geqslant b \\ \infty & x < b. \end{cases} \tag{2.6}$$

Therefore, the eigenfunctions have their support on the interval $[b, +\infty)$, where they satisfy the equation

$$\frac{d^2 y(x)}{dx^2} - \left(\frac{x^2}{4} + \epsilon\right) y(x) = 0. \tag{2.7}$$

Physically, for these wavefunctions to be meaningful they have to comply with two extra requirements:

(i) $y(x)$ must be square integrable;

(ii) $y(x)$ must be continuous on $\mathbb{R}$, i.e. $y(x) \to 0$ when $x \to b^+$.

Now, we shall determine such eigenfunctions together with their corresponding eigenvalues. Recall that (2.7) is one of the standard forms of the differential equation giving rise to the *parabolic cylinder functions* (see [12, p 686]) which include, as a particular case, the Hermite functions. The the most suitable form of the general solution of (2.7) for our present purposes is a linear combination of the so-called 'standard solutions' $U(\epsilon, x)$ and $V(\epsilon, x)$ (which in their turn can be expressed in terms of Whittaker functions). The power series expansion for such functions is the following [12, pp 686–7]:

$$\begin{aligned} U(\epsilon, x) = {} & \frac{\sqrt{\pi} e^{-x^2/4}}{2^{\frac{\epsilon}{2}+\frac{1}{4}} \Gamma(\frac{3}{4} + \frac{\epsilon}{2})} \left\{1 + \left(\epsilon + \frac{1}{2}\right) \frac{x^2}{2!} + \left(\epsilon + \frac{1}{2}\right)\left(\epsilon + \frac{5}{2}\right) \frac{x^4}{4!} + \cdots\right\} \\ & - \frac{\sqrt{\pi} e^{-x^2/4}}{2^{\frac{\epsilon}{2}-\frac{1}{4}} \Gamma(\frac{1}{4} + \frac{\epsilon}{2})} \left\{x + \left(\epsilon + \frac{3}{2}\right) \frac{x^3}{3!} + \left(\epsilon + \frac{3}{2}\right)\left(\epsilon + \frac{7}{2}\right) \frac{x^5}{5!} + \cdots\right\} \end{aligned} \tag{2.8}$$

$$\begin{aligned} V(\epsilon, x) = {} & \frac{\sqrt{\pi} \tan[(\frac{1}{4} + \frac{\epsilon}{2})\pi] e^{-x^2/4}}{2^{\frac{\epsilon}{2}+\frac{1}{4}} \Gamma(\frac{3}{4} + \frac{\epsilon}{2}) \Gamma(\frac{1}{2} - \epsilon)} \left\{1 + \left(\epsilon + \frac{1}{2}\right) \frac{x^2}{2!} + \left(\epsilon + \frac{1}{2}\right)\left(\epsilon + \frac{5}{2}\right) \frac{x^4}{4!} + \cdots\right\} \\ & + \frac{\sqrt{\pi} \cot[(\frac{1}{4} + \frac{\epsilon}{2})\pi] e^{-x^2/4}}{2^{\frac{\epsilon}{2}-\frac{1}{4}} \Gamma(\frac{1}{4} + \frac{\epsilon}{2}) \Gamma(\frac{1}{2} - \epsilon)} \{x + \cdots\}. \end{aligned} \tag{2.9}$$

In our context, the key property of these functions is their asymptotic behaviour for $x \to +\infty$ and $x \gg |\epsilon|$:

$$\begin{aligned} U(\epsilon, x) & \sim \frac{e^{-x^2/4}}{x^{\epsilon+1/2}} \left\{1 - \frac{(\epsilon + \frac{1}{2})(\epsilon + \frac{3}{2})}{2x^2} + \cdots\right\} \\ V(\epsilon, x) & \sim \frac{\sqrt{2/\pi} e^{+x^2/4}}{x^{-\epsilon+1/2}} \left\{1 + \frac{(\epsilon - \frac{1}{2})(\epsilon - \frac{3}{2})}{2x^2} + \cdots\right\}. \end{aligned} \tag{2.10}$$

Note that $V(\epsilon, x)$ diverges for $x \to +\infty$. As we are interested in a solution for the wavefunction $y(x)$ bounded in that limit, we must discard $V(\epsilon, x)$ and keep only the other independent solution $U(\epsilon, x)$.

The unnormalized solution of the Schrödinger equations (2.5) and (2.6) is obtained from the above discussion of (2.7), valid for $x \geqslant b$, and taking the value zero for $x < b$,

$$y(x) = \begin{cases} U(\epsilon, x) & x \geqslant b \\ 0 & x < b. \end{cases} \tag{2.11}$$

This gives the answer to the first requirement (i). Since this wavefunction must also be continuous (but not necessarily its first derivative) at the discontinuity of the potential, we must have

$$U(\epsilon, b) = 0 \tag{2.12}$$

for the second condition (ii) to be satisfied. Therefore, for every fixed value of b the zeros of equation (2.12) provide a discrete ensemble of solutions $\epsilon_n(b)$, $n = 0, 1, 2, \ldots$, that, taking into account the relation (2.3) between ϵ and the energy E, give us the energy levels of this problem $E_n(b)$. Let us now consider the simplest cases.

Case 1. If we take $b = 0$, from (2.12) and (2.8) we have

$$U(\epsilon, 0) = \frac{\sqrt{\pi}}{2^{\frac{\epsilon}{2}+\frac{1}{4}}\Gamma(\frac{3}{4}+\frac{\epsilon}{2})} = 0. \tag{2.13}$$

The solutions of this equation are the singularities of the gamma function in the denominator, that is

$$\frac{3}{4} + \frac{\epsilon_n}{2} = -n \qquad n = 0, 1, 2, \ldots \tag{2.14}$$

and taking into account (2.3), we obtain the following energy spectrum

$$E_n = -\hbar\omega\epsilon_n = \hbar\omega(2n + 1 + \tfrac{1}{2}) \qquad n = 0, 1, 2, \ldots. \tag{2.15}$$

Note that we recover only the odd energy levels of the harmonic oscillator, corresponding to the odd wavefunctions (those which are null at the origin).

Case 2. Another simple situation is obtained when we consider the limiting case $b \to -\infty$, in order to recover the spectrum of the harmonic oscillator in the whole real line. In this case, we have to carefully analyse the limit $\lim_{b\to-\infty} U(\epsilon, b)$. Hence, let us write (2.8) in terms of confluent hypergeometric functions [12]

$$U(\epsilon, x) = \frac{\sqrt{\pi}\mathrm{e}^{-x^2/4}}{2^{\frac{\epsilon}{2}+\frac{1}{4}}\Gamma(\frac{3}{4}+\frac{\epsilon}{2})}\,{}_1F_1\left(\frac{\epsilon}{2}+\frac{1}{4}; \frac{1}{2}; \frac{x^2}{2}\right) - x\frac{\sqrt{\pi}\mathrm{e}^{-x^2/4}}{2^{\frac{\epsilon}{2}-\frac{1}{4}}\Gamma(\frac{1}{4}+\frac{\epsilon}{2})}\,{}_1F_1\left(\frac{\epsilon}{2}+\frac{3}{4}; \frac{3}{2}; \frac{x^2}{2}\right). \tag{2.16}$$

Now, we take into account the asymptotic behaviour of the function ${}_1F_1(a; c; z)$ for z real and $z \to +\infty$ (see for example [13, p 278] or [12, p 508]):

$${}_1F_1(a; c; z) \sim \frac{\Gamma(c)}{\Gamma(a)}\frac{\mathrm{e}^z}{z^{c-a}}\left\{1 + \frac{(1-a)(c-a)}{1!z} + \cdots\right\}.$$

Using this result in (2.16), we have for $b \to -\infty$

$$U(\epsilon, b) \sim \frac{\pi 2^{1-\epsilon}}{|b|^{1/2-\epsilon}}\frac{\mathrm{e}^{|b|^2/4}}{\Gamma(\frac{1}{4}+\frac{\epsilon}{2})\Gamma(\frac{3}{4}+\frac{\epsilon}{2})}. \tag{2.17}$$

Therefore, the solutions of equation (2.12) are in this case the singularities of the two gamma functions that appear in the denominator of (2.17), that is

$$E_n = -\hbar\omega\epsilon_n = \hbar\omega(n + \tfrac{1}{2}) \qquad n = 0, 1, 2, \ldots \tag{2.18}$$

which is the complete spectrum of the harmonic oscillator, as expected.

In the two extreme cases that we considered before, the spectra that appear are equally spaced: one unit in (2.18) and two units in (2.15). For other values of b the situation is more complicated, and a numerical approximation of what is happening can be seen in figure 1. The interpretation of this figure is the following: when b takes the value $-\infty$, the eigenvalues of the energy are equidistant in one unit; as b approaches zero, the energy levels change accordingly in a continuous form, but are no longer equidistant; for $b = 0$ the energy levels are again equally spaced, but by two units; finally, as b takes increasingly positive values, the levels continue to grow, as well as the separation between them. The exact analytic form of those levels $E_n(b) = -\hbar\omega\epsilon_n(b)$ as a function of b cannot be simply obtained from (2.12).

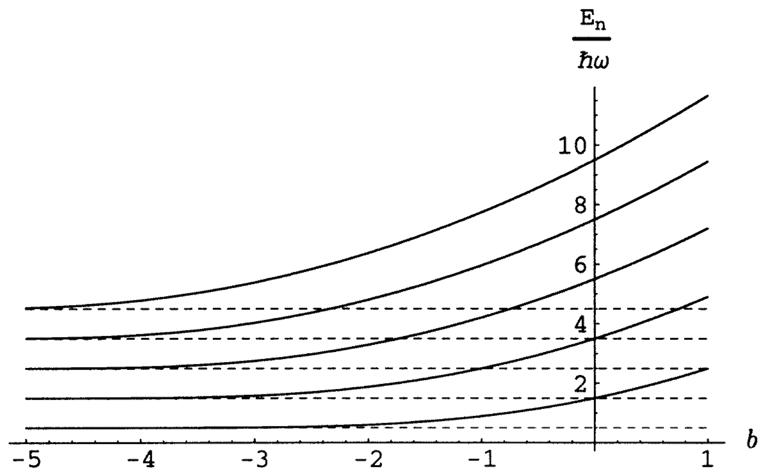

Figure 1. Energy levels of the oscillator with infinite barrier at b.

3. Factorization method for the oscillator

Let us now turn our attention towards the application of the factorization method to the oscillator. In that respect, we shall take [5] as a reference for the cases to follow. In that paper, limited to values of a parameter for which the new potentials are free from singularities, the one-parametric Abraham–Moses family of potentials [14] was derived. Our purpose here is to prove that the singular cases ignored in [5] can also be solved, establishing a relationship with the analysis carried out in the previous section.

3.1. First factorization: singular and nonsingular potentials

Going back to the notation used in section 2, the factorization of equation (2.7) is accomplished if we look for the most general function $\alpha(x)$ such that

$$Hy = \left(-\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + \frac{x^2}{4}\right) y \equiv \left[\left(\frac{\mathrm{d}}{\mathrm{d}x} + \alpha\right)\left(-\frac{\mathrm{d}}{\mathrm{d}x} + \alpha\right) + K\right] y = -\epsilon y \tag{3.1}$$

where K is a constant. Thus, $\alpha(x)$ must satisfy the Riccati equation

$$\frac{\mathrm{d}\alpha}{\mathrm{d}x} + \alpha^2 + K - \frac{x^2}{4} = 0. \tag{3.2}$$

As is well known such types of Ricatti equations are essentially equivalent to stationary Schrödinger equations, in this particular case with potential $x^2/4$ and eigenvalue K (see for example [15, p 333], or the chapter on the WKB method in [16]). We shall fix our attention on the two specific values for K considered in [5], any other can be dealt along the same lines [17].

First, let us take $K = -\frac{1}{2}$. For this choice a particular solution of (3.2) is $\alpha(x) = x/2$, corresponding to the usual lowering and raising operators $a, a^\dagger$. From here, the general solution can be immediately obtained:

$$\alpha_\delta(x) = \frac{x}{2} + \frac{\delta \mathrm{e}^{-x^2/2}}{1 + \delta \int_0^x \mathrm{e}^{-t^2/2}\,\mathrm{d}t} \qquad \delta \in \mathbb{R}. \tag{3.3}$$

If we call a_δ and $a_\delta^\dagger$ to the factor operators in (3.1)

$$a_\delta = \frac{\mathrm{d}}{\mathrm{d}x} + \alpha_\delta(x) = \frac{\mathrm{d}}{\mathrm{d}x} + \frac{x}{2} + \frac{\delta \mathrm{e}^{-x^2/2}}{1+\delta\int_0^x \mathrm{e}^{-t^2/2}\,\mathrm{d}t} \tag{3.4}$$

$$a_\delta^\dagger = -\frac{\mathrm{d}}{\mathrm{d}x} + \alpha_\delta(x) = -\frac{\mathrm{d}}{\mathrm{d}x} + \frac{x}{2} + \frac{\delta \mathrm{e}^{-x^2/2}}{1+\delta\int_0^x \mathrm{e}^{-t^2/2}\,\mathrm{d}t} \tag{3.5}$$

equation (3.1) can be rewritten as

$$Hy(x) = (a_\delta a_\delta^\dagger - \tfrac{1}{2})y(x) = -\epsilon y(x). \tag{3.6}$$

Note that $a, a^\dagger$ are recovered just by taking the value $\delta = 0$. If we consider the product of the operators a_δ and $a_\delta^\dagger$ in the reverse order, what we find is a new Hamiltonian H_δ defined by

$$H_\delta \equiv a_\delta^\dagger a_\delta - \frac{1}{2} = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} - \alpha' + \alpha^2 - \frac{1}{2} \tag{3.7}$$

and whose explicit form is

$$H_\delta = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V_\delta(x) = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{x^2}{4} - 1 + \frac{2\delta x \mathrm{e}^{-x^2/2}}{1+\delta\int_0^x \mathrm{e}^{-t^2/2}\,\mathrm{d}t} + \frac{2\delta^2 \mathrm{e}^{-x^2}}{[1+\delta\int_0^x \mathrm{e}^{-t^2/2}\,\mathrm{d}t]^2}. \tag{3.8}$$

This one-parametric class of Hamiltonians H_δ may have singularities depending on the existence of solutions for the equation

$$1+\delta\int_0^x \mathrm{e}^{-t^2/2}\,\mathrm{d}t \equiv 1 + \delta\sqrt{\frac{\pi}{2}}\mathrm{erf}\left(\frac{x}{\sqrt{2}}\right) = 0. \tag{3.9}$$

Because of the properties of the error function, (3.9) has no solution if the parameter δ is such that $|\delta| < \sqrt{2/\pi}$. Those are the nonsingular cases; for other real values of δ the Hamiltonian H_δ is singular. The typical form of the nonsingular potentials $V_\delta(x)$ can be seen in [18]. Some illustrative cases for singular potentials are shown in figure 2.

The spectrum of the Hamiltonians H, H_δ are closely linked for any value of δ. From (3.6) and (3.7) it is clear that the following relation holds

$$H_\delta a_\delta^\dagger = a_\delta^\dagger H \tag{3.10}$$

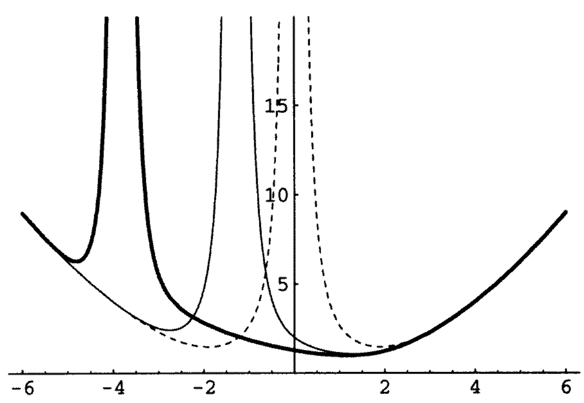

Figure 2. Singular potential $V_\delta(x)$ for $\delta = 0.798$ (full), $\delta = 1$ (light), and $\delta = 1000$ (broken).

as well as its Hermitian conjugate

$$H a_\delta = a_\delta H_\delta. \tag{3.11}$$

Relations (3.10) and (3.11) imply that if $\psi(x)$ is an eigenstate of H (H_δ) with eigenvalue λ, then $a_\delta^\dagger \psi(x)$ ($a_\delta \psi(x)$) will be an eigenstate of H_δ (H) having the same eigenvalue.

This is a purely algebraic relationship that does not guarantee that if $a_\delta^\dagger$ acts on the states of the Hilbert space associated with H, the images will belong to the Hilbert space of H_δ. In other words, it does not take into account important details such as the presence of singularities or the boundary conditions. These topics have been considered in [5] for nonsingular potentials ($|\delta| < \sqrt{2/\pi}$), and the results can be summarized as follows.

(1) If ψ is an eigenfunction of H that is square integrable on the whole real line, then the eigenfunction $a_\delta^\dagger \psi$ of H_δ is also in $\mathcal{L}^2(\mathbb{R})$.

(2) Reciprocally, if ϕ is an eigenfunction of H_δ in $\mathcal{L}^2(\mathbb{R})$, then $a_\delta \phi$ is eigenfunction of H also in $\mathcal{L}^2(\mathbb{R})$. The equation $a_\delta \phi = 0$ determines the square integrable ground state of H_δ. The Hamiltonians H and H_δ are isospectral except for the ground state of H_δ corresponding to the eigenvalue $-\frac{1}{2}$.

The singular cases not touched in [5] will be our main concern from now on. First of all, note that if we are in the singular domain ($|\delta| > \sqrt{2/\pi}$), then for every fixed value of δ there is exactly one finite value of x solution of (3.9), let us call it x_δ. The behaviour of the singular potential in a small neigbourhood of the singularity x_δ is

$$V_\delta(x) \approx \frac{2}{(x - x_\delta)^2}. \tag{3.12}$$

Physically this implies that if a particle is initially to the r.h.s. of the singularity $x > x_\delta$, then it is going to be confined to that region in the future, because the probability of going through the barrier towards the left of x_δ is null (see [19, 7, p 359]). In conclusion the two regions $x > x_\delta$ and $x < x_\delta$ are physically disconnected so that the eigenfunctions for the singular potentials must be computed independently for each region. In the following we shall restrict ourselves to the r.h.s. $x > x_\delta$, but obviously similar considerations apply to the l.h.s. The eigenfunctions describing acceptable physical states in this region must satisfy

$$\begin{aligned} &\text{(a)} \qquad H_\delta \phi(x) = -\epsilon \phi(x) \qquad x \geqslant x_\delta \\ &\text{(b)} \qquad \phi(x) \in \mathcal{L}^2([x_\delta, +\infty)) \\ &\text{(c)} \qquad \phi(x) \to 0 \qquad \text{if } x \to x_\delta^+. \end{aligned} \tag{3.13}$$

We can think that $\phi(x)$ is defined on the whole real line $\mathbb{R}$ assuming $\phi(x) = 0$ for $x < x_\delta$.

From the above discussion it is clear that if $\psi(x)$ is a (square integrable) eigenfunction of the oscillator Hamiltonian, when we take a value δ corresponding to a singular potential, the wavefunction $a_\delta^+ \psi(x)$ will not satisfy the previous conditions, mainly because of (c). Thus, the spectrum of the singular Hamiltonians H_δ have nothing to do with the oscillator one. However, we shall show next that the singular H_δ are directly related to the harmonic oscillator plus an infinite barrier of section 2.

3.2. The singular potentials in depth

In order to prove the connection between the eigenvalue equations (2.5), (2.6) and (3.13), note that in the region $[b, +\infty)$ the first one has the same differential expression as for the oscillator potential as can be seen in (2.7). Therefore, on that interval we can apply the development of the previous section; in particular, for any singular H_δ the connections between the eigenfunctions derived from (3.10) and (3.11) are still valid.

Now, the eigenfunctions for the oscillator plus a barrier set at the point b corresponding to energy E_n are given by (2.11) and (2.12), and henceforth will be denoted by $\{y_n^{(b)}(x)\}_{n=0}^{+\infty}$. Thus,

$$y_n^{(b)}(x) = \begin{cases} U(\epsilon_n(b), x) & x \geqslant b \\ 0 & x \leqslant b. \end{cases} \tag{3.14}$$

If we act on these eigenfunctions with an operator $a_\delta^\dagger$ (3.5) that presents a singularity at x_δ, we obtain

$$a_\delta^\dagger y_n^{(b)}(x) = \begin{cases} \left[-\dfrac{\mathrm{d}}{\mathrm{d}x} + \dfrac{x}{2} + \dfrac{\delta \mathrm{e}^{-x^2/2}}{1+\delta\int_0^x \mathrm{e}^{-t^2/2}\,\mathrm{d}t}\right] U(\epsilon_n(b), x) & x \geqslant b \\ 0 & x < b. \end{cases} \tag{3.15}$$

This is an eigenfunction of H_δ corresponding to the same eigenvalue. However, we must now deal with the problems related to its boundary behaviour.

First, the resulting function (3.15) may have a singularity at x_δ unless at this point there is a zero of $U(\epsilon_n(b), x)$. This fact can be guaranteed if we take precisely $b = x_\delta$, the solution of (3.9). Therefore, we are connecting the harmonic oscillator potential plus a barrier at b with a singular potential whose singularity is set at the same point b. The value of δ satisfying this condition will be referred to as $\delta(b)$. Once we have fixed such a value $\delta(b)$, we can write the action of the creation operator (that we will denote hereafter $a_{\delta(b)}^\dagger$) on the eigenfunctions $y_n^{(b)}(x)$ as $f_n^{(b)}(x) := a_{\delta(b)}^\dagger y_n^{(b)}(x)$.

This choice of $\delta(b)$ will assure that $f_n^{(b)}(x)$ is square integrable, but a second important point to be verified is whether the functions $f_n^{(b)}(x)$ are really continuous at $x = b$, as one would expect for a physical wavefunction. This fact is easily checked, because using (2.12) the local behaviour of $f_n^{(b)}(x)$ for $x \geqslant b$ is given by a Taylor series

$$U(\epsilon_n(b), x) = \mu_n(x-b) + \rho_n(x-b)^2 + \cdots$$

and therefore

$$\left[-\frac{\mathrm{d}}{\mathrm{d}x} + \frac{x}{2} + \frac{\mathrm{e}^{-x^2/2}}{\int_b^x \mathrm{e}^{-t^2/2}\,\mathrm{d}t}\right] U(\epsilon_n(b), x) \approx \left[-\frac{\mathrm{d}}{\mathrm{d}x} + \frac{x}{2} + \frac{1}{(x-b)}\right](\mu_n(x-b)+\cdots)$$
$$\approx \frac{b\mu_n - 2\rho_n}{2}(x-b) + \cdots.$$

However, if we act with $a_{\delta(b)}$ on this result, bear in mind that $H = a_{\delta(b)} a_{\delta(b)}^\dagger - \frac{1}{2}$, we conclude that $b\mu_n - 2\rho_n = 0$. Hence, near the singularity we have,

$$f_n^{(b)}(x) \approx \kappa(x-b)^2 + \cdots$$

and the eigenfunctions have a good behaviour in the limit $x \to b^+$.

Finally, observe that the whole set of eigenfunctions and eigenvalues of $H_{\delta(b)}$ are precisely those already mentioned:

$$f_n^{(b)}(x) = a_{\delta(b)}^\dagger y_n^{(b)}(x) \qquad \lambda_n^{(b)} = -\epsilon_n(b) \qquad n = 0, 1, 2, \ldots.$$

The completeness of this set is assured by the fact that the state annihilated by the operator $a_{\delta(b)}$ is

$$\hat{f}_0^{(b)}(x) = \begin{cases} K_0 \dfrac{\mathrm{e}^{-x^2/4}}{\int_b^x \mathrm{e}^{-t^2/2}\,\mathrm{d}t} & x \geqslant b \\ 0 & < b \end{cases} \tag{3.16}$$

which has a singularity at $x = b$, and therefore cannot be an admissible wavefunction for the problem under study.

3.3. Second type of factorization and new singular potentials

We considered before a class of factorizations of the oscillator coming from the expression $H = aa^\dagger - \frac{1}{2}$ in the notation of [5], but in the same way there is another symmetric factorization starting from $H = a^\dagger a + \frac{1}{2}$ corresponding to $K = \frac{1}{2}$ in terms of (3.1). The latter was not taken into account in [5] probably because of the appearence of singular potentials. As we are dealing just with those singular cases, it seems quite natural to explore it under our viewpoint. Therefore, let us write the harmonic oscillator Hamiltonian factorized in the form

$$Hy = \left(-\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + \frac{x^2}{4}\right) y \equiv \left[\left(-\frac{\mathrm{d}}{\mathrm{d}x} + \beta\right)\left(\frac{\mathrm{d}}{\mathrm{d}x} + \beta\right) + \frac{1}{2}\right] y = -\epsilon y. \tag{3.17}$$

The general solution for $\beta_\gamma(x)$ turns out to be

$$\beta_\gamma(x) = \frac{x}{2} + \frac{\gamma \mathrm{e}^{x^2/2}}{1 - \gamma \int_0^x \mathrm{e}^{t^2/2}\,\mathrm{d}t} \qquad \gamma \in \mathbb{R}. \tag{3.18}$$

Denoting by a_γ and $a_\gamma^\dagger$ the new creation and annihilation operators

$$a_\gamma = \frac{\mathrm{d}}{\mathrm{d}x} + \beta_\gamma(x) = \frac{\mathrm{d}}{\mathrm{d}x} + \frac{x}{2} + \frac{\gamma \mathrm{e}^{x^2/2}}{1 - \gamma \int_0^x \mathrm{e}^{t^2/2}\,\mathrm{d}t} \tag{3.19}$$

$$a_\gamma^\dagger = -\frac{\mathrm{d}}{\mathrm{d}x} + \beta_\gamma(x) = -\frac{\mathrm{d}}{\mathrm{d}x} + \frac{x}{2} + \frac{\gamma \mathrm{e}^{x^2/2}}{1 - \gamma \int_0^x \mathrm{e}^{t^2/2}\,\mathrm{d}t} \tag{3.20}$$

the equation (3.17) can be written as

$$Hy(x) = (a_\gamma^\dagger a_\gamma + \tfrac{1}{2})y(x) = -\epsilon y(x). \tag{3.21}$$

We obtain a uniparametric family of new Hamiltonians by writing the product

$$H_\gamma \equiv a_\gamma a_\gamma^\dagger + \frac{1}{2} = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \beta_\gamma' + \beta_\gamma^2 + \frac{1}{2}. \tag{3.22}$$

The explicit form of H_γ is

$$H_\gamma = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V_\gamma(x) = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{x^2}{4} + 1 + \frac{2\gamma x \mathrm{e}^{x^2/2}}{1 - \gamma \int_0^x \mathrm{e}^{t^2/2}\,\mathrm{d}t} + \frac{2\gamma^2 \mathrm{e}^{x^2}}{\left[1 - \gamma \int_0^x \mathrm{e}^{t^2/2}\,\mathrm{d}t\right]^2}. \tag{3.23}$$

An important detail that establishes a difference between this second factorization and the first one is that now the potentials $V_\gamma(x)$ always present a singularity, as long as $\gamma \neq 0$. This is due to the fact that the equation

$$\int_0^x \mathrm{e}^{t^2/2}\,\mathrm{d}t = \frac{1}{\gamma} \tag{3.24}$$

always has a unique solution x_γ for every fixed non-null value of γ. The behaviour of this singular potential in a neigbourhood of the singularity x_γ is also $V_\gamma(x) \approx 2/(x - x_\gamma)^{-2}$. Some typical examples can be seen in figure 3.

Instead of the intertwining relations (3.10) and (3.11), we now have

$$H_\gamma a_\gamma = a_\gamma H \qquad H a_\gamma^\dagger = a_\gamma^\dagger H_\gamma. \tag{3.25}$$

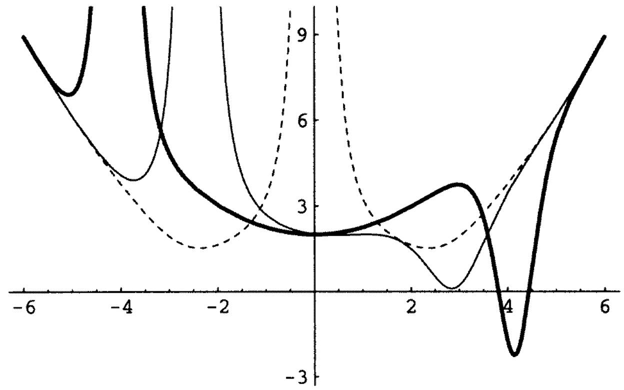

Figure 3. Singular potential $V_\gamma(x)$ for $\gamma = -0.001$ (full), $\gamma = 1$ (light), and $\gamma = 100$ (broken).

Therefore, if ψ is an eigenvector of H with eigenvalue λ, then $a_\gamma\psi$ is an eigenvector of H_γ with eigenvalue λ. Let us consider the action of the operator a_γ on the function $y_n^{(b)}(x)$ written in (3.14).

$$a_\gamma y_n^{(b)}(x) = \begin{cases} \left[\dfrac{\mathrm{d}}{\mathrm{d}x} + \dfrac{x}{2} - \dfrac{\mathrm{e}^{x^2/2}}{\int_{x_\gamma}^{x} \mathrm{e}^{t^2/2}\,\mathrm{d}t}\right] U(\epsilon_n(b), x) & x \geqslant b \\ 0 & x \leqslant b. \end{cases}$$

Following the same reasoning as that developed for the previous factorization, in order for this function not to have a singularity at x_γ we must take $b = x_\gamma$. The functions $a_{\gamma(b)} y_n^{(b)}(x)$ are just the eigenfunctions of $H_{\gamma(b)}$, with eigenvalues $-\epsilon_n(b)$, that constitute the whole spectrum of $H_{\gamma(b)}$.

4. Concluding remarks

Usually, the factorization method is applied to connect a regular Hamiltonian to another regular one, or a singular to a singular one. Here we have shown with one simple example how the regular to singular situation must be handled.

There is just one case where the singular potentials can be solved analytically, this is when the barrier (or the singularity point) is set at the origin. Then, the symmetry conditions allows us to solve the problem completely. This situation is particularly interesting from the physical viewpoint, since the x variable can be interpreted as the radial coordinate in a three-dimensional spherical problem, so that $x \geqslant 0$. In such circumstances sections 2 and 3 provide us with two classes of singular potentials that are exactly solvable (note that even in this case the singularity does not have a centrifugal character [20]). This variety of new potentials can be of interest when one tries to reproduce phenomenological data by adjusting the parameters that characterize the class of solvable potentials.

Finally we wish to point out that although in this work we have restricted ourselves to the oscillator potential, the same arguments can be applied to any of the known factorizable potentials, for instance Morse, Coulomb, etc. (Some work in this direction is in progress.) Also, we have considered here just one singularity point, but it is obvious that, with slight modifications, our method can be extended when more singular points appear, for instance when the system is confined to a bounded interval of $\mathbb{R}$.

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